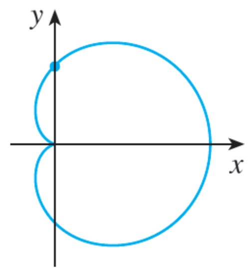
Exercise 29

Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2, \quad \left(0, \frac{1}{2}\right), \quad \text{(cardioid)}$$



Solution

The aim is to evaluate y' at $x = 0$ and $y = 1/2$ in order to find the slope there. Differentiate both sides of the given equation with respect to x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}[(2x^2 + 2y^2 - x)^2]$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 2(2x^2 + 2y^2 - x) \cdot \frac{d}{dx}(2x^2 + 2y^2 - x)$$

$$(2x) + \left[2y \cdot \frac{d}{dx}(y)\right] = 2(2x^2 + 2y^2 - x) \cdot \left[2\frac{d}{dx}(x^2) + 2\frac{d}{dx}(y^2) - \frac{d}{dx}(x)\right]$$

$$(2x) + (2yy') = 2(2x^2 + 2y^2 - x) \cdot \left[2(2x) + 2(2yy') - (1)\right]$$

$$2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1)$$

$$2x + 2yy' = 16x^3 + 16x^2yy' - 4x^2 + 16xy^2 + 16y^3y' - 4y^2 - 8x^2 - 8xyy' + 2x$$

$$2yy' = 16x^3 + 16x^2yy' - 12x^2 + 16xy^2 + 16y^3y' - 4y^2 - 8xyy'$$

$$yy' = 8x^3 + 8x^2yy' - 6x^2 + 8xy^2 + 8y^3y' - 2y^2 - 4xyy'$$

Solve for y'.

$$(y - 8x^2y - 8y^3 + 4xy)y' = 8x^3 - 6x^2 + 8xy^2 - 2y^2$$

$$y' = \frac{8x^3 - 6x^2 + 8xy^2 - 2y^2}{y - 8x^2y - 8y^3 + 4xy}$$

Evaluate y' at $x = 0$ and $y = 1/2$.

$$y'\left(0, \frac{1}{2}\right) = \frac{8(0)^3 - 6(0)^2 + 8(0)\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right) - 8(0)^2\left(\frac{1}{2}\right) - 8\left(\frac{1}{2}\right)^3 + 4(0)\left(\frac{1}{2}\right)} = 1$$

Therefore, the equation of the tangent line to the curve represented by $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$ at $\left(0, \frac{1}{2}\right)$ is

$$y - \frac{1}{2} = 1(x - 0).$$

Below is a graph of the curve and the tangent line at $\left(0, \frac{1}{2}\right)$.